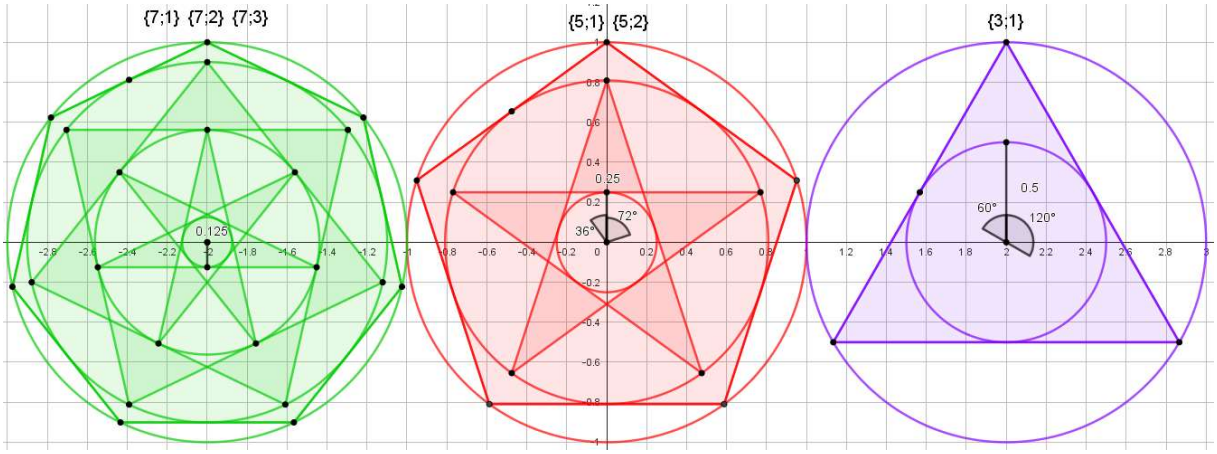
3.1 Flächenverhältnisse bei Umkreis – Kaskade von Sternen – Inkreis, S.88f

1.Behauptung:
n ungerade, $n \geq 3$
K_u = Umkreis (r = 1), K_{ij} = Inkreis j mit Radius r_j, k := (n-1)/2, {n;1} = n-Eck, {n;j} n-Sterne
Kaskade: K_u – {n;1} – K_{i1} – {n;2} – K_{i2} – … – {n;k} – K_{ik}

⇒ Flächenverhältniss $K_u : K_k = 2^{n-1}$

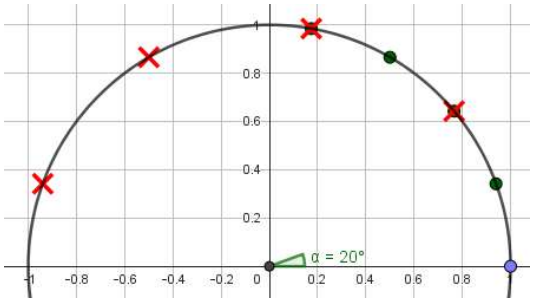


Beweis: z.z. $r_k = 1/2^k$
Der Mittelpunktswinkel beim n-Eck ist 360°/n zwischen benachbarten Ecken.
⇒ $r_1 = \cos(360°/(2n)) = \cos(\alpha)$ mit $\alpha := 360°/(2n)$
⇒ $r_2 = r_1 \cos(2\alpha) = \cos(\alpha)*\cos(2\alpha)$ …
⇒ $r_k = \cos(\alpha)*\cos(2\alpha)*...*\cos(k\alpha)$, mit $k\alpha = \frac{n-1}{2} * \frac{360°}{2n} = \frac{n-1}{n} * 90° < 90°$

wegen $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ ist $\cos(\alpha) = \frac{\sin(2\alpha)}{2\sin(\alpha)}$

⇒ $r_k = \prod_{j=1}^{k} \cos(j\alpha) = \frac{\sin(2\alpha)\sin(4\alpha)...\sin(2k\alpha)}{2^k * \sin(\alpha)\sin(2\alpha)...\sin(k\alpha)} = 1/2^k$

denn für $90° < 2x < 180°$ ist $\sin(2x) = \sin(180°-2x)$



z.B. für n = 9, k = 4, α = 20°:
$\sin(120°) = \sin(60°)$, $\sin(160°) = \sin(20°)$

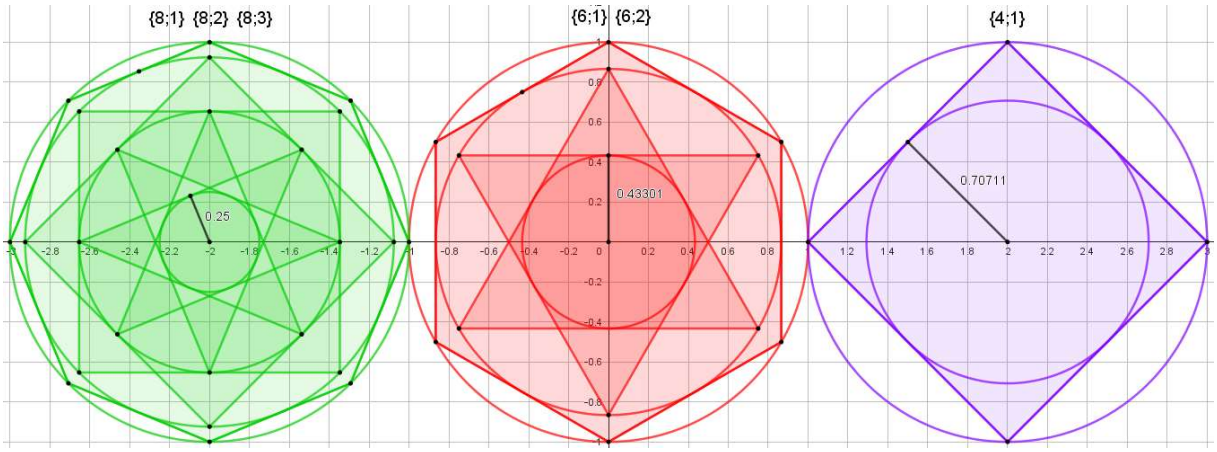
Also geometrisches Mittel von
$\cos(\alpha)*\cos(2\alpha)*...*\cos(k\alpha) = 1/2$

2.Behauptung:

n gerade, $n \geq 4$

K_u = Umkreis (r = 1), K_{ij} = Inkreis j mit Radius r_j, k := n/2-1, {n;1} = n-Eck, {n;j} n-Sterne

Kaskade: K_u – {n;1} – K_{i1} – {n;2} – K_{i2} – … – {n;k} – K_{ik}



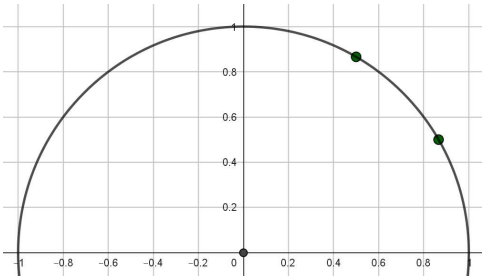
⇒ Flächenverhältniss $K_u : K_k = 2^{n-1}/n$

Beweis: z.z. $r_k^2 = \dfrac{n}{2^{n-1}}$

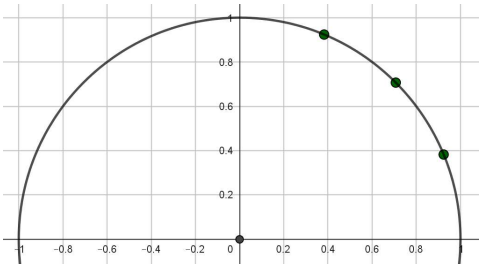
$\alpha := 360°/(2n)$ ⇒ $r_k = \cos(\alpha)*\cos(2\alpha)*...*\cos(k\alpha) = \sin(k\alpha)*\sin((k-1)\alpha)*...\sin(\alpha)$

da z.B.

n = 6, $r_i = \cos(30°)*\cos(60°)$
$= \sin(60°)*\sin(30°)$



n = 8, $r_i = \cos(22{,}5°)*\cos(45°)*\cos(67{,}5°)$
$= \sin(67{,}5°)*\sin(45°)*\sin(22{,}5°)$



allgemein:

$r_k = \sin(\alpha)*\sin(2\alpha)* \dots *\sin(k\alpha)$

$r_k^2 = \sin(\alpha)*\sin(2\alpha)* \dots \sin(k\alpha) * 1 * \sin(k\alpha)*\sin((k-1)\alpha)* \dots *\sin(\alpha)$

$r_k^2 = \sin(\alpha)*\sin(2\alpha)* \dots \sin(k\alpha) * \sin((k+1)\alpha) * \sin((k+2)\alpha)*\sin((k+3)k\alpha)* \dots *\sin((2k+1)\alpha)$

d.h. n-1 Faktoren

wegen $\sin((k+1)\alpha) = \sin(n/2*180°/n) = \sin(90°) = 1$

und $\sin(90°+j\alpha) = \sin(90°-j\alpha)$, also $\sin((k+1+j)\alpha) = \sin((k+1-j)\alpha)$

somit: $r_k^2 = \prod_{j=1}^{n-1} \sin(180°/n*j) = n/2^{n-1}$ nach stack exchange (*)

https://math.stackexchange.com/questions/70231/how-to-prove-those-curious-identities/70244#70244 : $\prod_{j=1}^{n-1} \sin(180°/n*j) = n/2^{n-1} = r_i^2$ (*)

Beweis:

$$\lim_{z=1} \frac{z^n - 1}{z - 1} = n \tag{1a}$$

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} (z - e^{2\pi i k/n}) \tag{1b}$$

$$|1 - e^{i2k\pi/n}| = |2\sin(k\pi/n)| \tag{1c}$$

Combining $(1a)$, $(1b)$, and $(1c)$, we get

$$2^{n-1} \prod_{k=1}^{n-1} \sin(k\pi/n) = n$$

since everything is positive.

(1a)

geometrische Reihe: $(z^n-1)/(z-1) = 1+z+z^2+...+z^{n-1} \rightarrow n$ für $z \rightarrow 1$

(1a) follows from L'Hôpital.

(1b) is just noting that the roots of z^n-1 are the nth roots of unity.

Dividing by $z-1$ removes the root at 1 and so we're left with the other $n-1$ roots $\{e^{2\pi ik/n}: 1 \le k \le n-1\}$.

Denote $w = e^{i\pi/n}$. We have

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \prod_{k=1}^{n-1} \frac{w^k - w^{-k}}{2i} = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{w^k}{i}(1 - w^{-2k})$$

Since we have

$$\sum_{k=0}^{n-1} x^k = \prod_{k=1}^{n-1} (x - w^{2k})$$

Setting $x = 1$ yields

$$\prod_{k=1}^{n-1} (1 - w^{2k}) = n$$

So we get

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \frac{w^{n(n-1)/2}}{i^{n-1}} = \frac{i^{n-1}}{i^{n-1}} \frac{n}{2^{n-1}} = \frac{n}{2^{n-1}}$$